OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS 4727

Further Pure Mathematics 3

Specimen Paper

Additional materials:
Answer booklet
Graph paper
List of Formulae (MF 1)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures, unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphic calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
- **You are reminded of the need for clear presentation in your answers.**

This question paper consists of 3 printed pages and 1 blank page.

1 Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = x,$$

giving y in terms of x in your answer. [5]

2 The set $S = \{a, b, c, d\}$ under the binary operation $*$ forms a group G of order 4 with the following operation table.

$*$	a	b	c	d
a	d	a	b	c
b	a	b	c	d
c	b	c	d	a
d	c	d	a	b

(i) Find the order of each element of G. [3]

(ii) Write down a proper subgroup of G. [1]

(iii) Is the group G cyclic? Give a reason for your answer. [1]

(iv) State suitable values for each of a, b, c and d in the case where the operation $*$ is multiplication of complex numbers. [1]

3 The planes Π_1 and Π_2 have equations $\mathbf{r}.(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = 1$ and $\mathbf{r}.(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 3$ respectively. Find

(i) the acute angle between Π_1 and Π_2, correct to the nearest degree, [4]

(ii) the equation of the line of intersection of Π_1 and Π_2, in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. [4]

4 In this question, give your answers exactly in polar form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

(i) Express $4((\sqrt{3}) - \mathrm{i})$ in polar form. [2]

(ii) Find the cube roots of $4((\sqrt{3}) - \mathrm{i})$ in polar form. [4]

(iii) Sketch an Argand diagram showing the positions of the cube roots found in part **(ii)**. Hence, or otherwise, prove that the sum of these cube roots is zero. [3]

5 The lines l_1 and l_2 have equations

$$\frac{x-5}{1} = \frac{y-1}{-1} = \frac{z-5}{-2} \quad \text{and} \quad \frac{x-1}{-4} = \frac{y-11}{-14} = \frac{z-2}{2}.$$

(i) Find the exact value of the shortest distance between l_1 and l_2. [5]

(ii) Find an equation for the plane containing l_1 and parallel to l_2 in the form $ax + by + cz = d$. [4]

6 The set S consists of all non-singular 2×2 real matrices $\mathbf{A}$ such that $\mathbf{AQ} = \mathbf{QA}$, where

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(i) Prove that each matrix $\mathbf{A}$ must be of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. [4]

(ii) State clearly the restriction on the value of a such that $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is in S. [1]

(iii) Prove that S is a group under the operation of matrix multiplication. (You may assume that matrix multiplication is associative.) [5]

7 **(i)** Prove that if $z = \mathrm{e}^{\mathrm{i}\theta}$, then $z^n + \frac{1}{z^n} = 2\cos n\theta$. [2]

(ii) Express $\cos^6 \theta$ in terms of cosines of multiples of θ, and hence find the exact value of

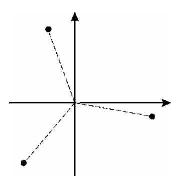
$$\int_0^{\frac{1}{3}\pi} \cos^6 \theta \, \mathrm{d}\theta.$$ [8]

8 **(i)** Find the value of the constant k such that $y = kx^2\mathrm{e}^{-2x}$ is a particular integral of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 2\mathrm{e}^{-2x}.$$ [4]

(ii) Find the solution of this differential equation for which $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$. [7]

(iii) Use the differential equation to determine the value of $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ when $x = 0$. Hence prove that $0 < y \leqslant 1$ for $x \geqslant 0$. [4]

Q	Answer	Mark		Guidance
1	Integrating factor is $e^{\int -x^{-1}\,dx} = e^{-\ln x} = \frac{1}{x}$	M1		For finding integrating factor
		A1		For correct simplified form
	$\frac{d}{dx}\left(\frac{y}{x}\right) = 1 \Rightarrow \frac{y}{x} = \int 1\,dx \Rightarrow y = x^2 + cx$	M1		For using integrating factor correctly
		B1		For arbitrary constant introduced correctly
		A1	5	For correct answer in required form
			[5]	
2 (i)	b is the identity and so has order 1	B1		For identifying b as the identity element
	$d * d = b$, so d has order 2	B1		For stating the order of d is 2
	$a * a = c * c = d$, so a and c each have order 4	B1	3	For both orders stated
(ii)	$\{b, d\}$	B1	1	For stating this subgroup
(iii)	G is cyclic because it has an element of order 4	B1	1	For correct answer with justification
(iv)	$b = 1, d = -1, a = \mathrm{i}, c = -\mathrm{i}$ (or *vice versa* for a, c)	B1	1	For all four correct values
			[6]	
3 (i)	Normals are $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$	B1		For identifying both normal vectors
	Acute angle is $\cos^{-1}\left(\frac{\lvert 2-4-2 \rvert}{3\times 3}\right) \approx 64°$	M1		For using the scalar product of the normals
		M1		For completely correct process for the angle
		A1	4	For correct answer
(ii)	Direction of line is $(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \times (2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$,	M1		For using vector product of normals
	i.e. $-2\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$	A1		For correct vector for **b**
	$x - 2y + 2z = 1,\ 2x + 2y - z = 3 \Rightarrow 3x + z = 4$,			
	so a common point is $(1, 1, 1)$, for example	M1		For complete method to find a suitable **a**
	Hence line is $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(-2\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$	A1	4	For correct equation of line
				(Other methods are possible)
			[8]	
4 (i)	$4\left((\sqrt{3}) - \mathrm{i}\right) = 8e^{-\frac{1}{6}\pi\mathrm{i}}$	B1		For $r = 8$
		B1	2	For $\theta = -\frac{1}{6}\pi$
(ii)	One cube root is $2e^{-\frac{1}{18}\pi\mathrm{i}}$	B1√		For modulus and argument both correct
	Others are found be multiplying by $e^{\pm\frac{2}{3}\pi\mathrm{i}}$	M1		For multiplication by either cube root of 1 (or equivalent use of symmetry)
	Giving $2e^{\frac{11}{18}\pi\mathrm{i}}$ and $2e^{-\frac{13}{18}\pi\mathrm{i}}$	A1		For either one of these roots
		A1	4	For both correct
(iii)		B1√		For correct diagram from their **(ii)**
	The roots have equal modulus and args differing by $\frac{2}{3}\pi$, so adding them geometrically makes a	M1		For geometrical interpretation of addition
	closed equilateral triangle; i.e. sum is zero	A1	3	For a correct proof (or via components, etc)
			[9]	

5	**(i)**	$(\mathbf{i}-\mathbf{j}-2\mathbf{k})\times(-4\mathbf{i}-14\mathbf{j}+2\mathbf{k})=-30\mathbf{i}+6\mathbf{j}-18\mathbf{k}$	M1	For vector product of direction vectors
		So common perp is parallel to $5\mathbf{i}-\mathbf{j}+3\mathbf{k}$	A1	For correct vector for common perp
		$(5\mathbf{i}+\mathbf{j}+5\mathbf{k})-(\mathbf{i}+11\mathbf{j}+2\mathbf{k})=4\mathbf{i}-10\mathbf{j}+3\mathbf{k}$	B1	For calculating the difference of positions
		$d=\frac{\lvert(4\mathbf{i}-10\mathbf{j}+3\mathbf{k}).(5\mathbf{i}-\mathbf{j}+3\mathbf{k})\rvert}{\lvert 5\mathbf{i}-\mathbf{j}+3\mathbf{k}\rvert}=\frac{39}{\sqrt{35}}$	M1	For calculation of the projection
			A1 **5**	For correct exact answer
	(ii)	Normal vector for plane is $5\mathbf{i}-\mathbf{j}+3\mathbf{k}$	B1√	For stating or using the normal vector
		Point on plane is $5\mathbf{i}+\mathbf{j}+5\mathbf{k}$	B1	For using any point of l_1
		Equation is $5x-y+3z=25-1+15$	M1	For using relevant direction and point
		i.e. $5x-y+3z=39$	A1 **4**	For a correct equation
			9	
6	**(i)**	$\mathbf{AQ}=\mathbf{QA}\Rightarrow\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}$	M1	For considering $\mathbf{AQ}=\mathbf{QA}$ with general **A**
		i.e. $\begin{pmatrix}a&a+b\\c&c+d\end{pmatrix}=\begin{pmatrix}a+c&b+d\\c&d\end{pmatrix}$	A1	For correct simplified equation
		Hence $a=a+c$ and $a+b=b+d$	M1	For equating corresponding entries
		i.e. $c=0$ and $d=a$	A1 **4**	For complete proof
	(ii)	To be non-singular, $a\neq 0$	B1 **1**	For stating that a is non-zero
	(iii)	Identity is $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ as usual, since this is in S	B1	For justifying the identity correctly
		Inverse of $\begin{pmatrix}a&b\\0&a\end{pmatrix}$ is $\begin{pmatrix}1/a&-b/a^2\\0&1/a\end{pmatrix}$, as $a\neq 0$	B1	For statement of correct inverse
			B1	For justification via non-zero a
		$\begin{pmatrix}a_1&b_1\\0&a_1\end{pmatrix}\begin{pmatrix}a_2&b_2\\0&a_2\end{pmatrix}=\begin{pmatrix}a_1a_2&a_1b_2+b_1a_2\\0&a_1a_2\end{pmatrix}$	M1	For considering a general product
		This is in S, since $a_1a_2\neq 0$, so all necessary group properties are shown	A1 **5**	For complete proof
			10	
7	**(i)**	$z^n=\cos n\theta+\mathrm{i}\sin n\theta$	B1	For applying de Moivre's theorem
		$z^{-n}=\cos n\theta-\mathrm{i}\sin n\theta$, hence $z^n+z^{-n}=2\cos n\theta$	B1 **2**	For complete proof
	(ii)	$2^6\cos^6\theta=(z+z^{-1})^6$	M1	For considering $(z+z^{-1})^6$
		$=(z^6+z^{-6})+6(z^4+z^{-4})+15(z^2+z^{-2})+20$	M1	For expanding and grouping terms
		$=2\cos 6\theta+12\cos 4\theta+30\cos 2\theta+20$	A1	For correct substitution of multiple angles
		Hence $\cos^6\theta=\frac{1}{32}(\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10)$	A1	For correct answer
		Integral is $\frac{1}{32}\left[\frac{1}{6}\sin 6\theta+\frac{3}{2}\sin 4\theta+\frac{15}{2}\sin 2\theta+10\theta\right]_0^{\frac{1}{3}\pi}$	M1	For integrating multiple angle expression
			A1√	For correct terms
		$=\frac{1}{32}\left(\frac{1}{6}\times 0+\frac{3}{2}\times(-\frac{1}{2}\sqrt{3})+\frac{15}{2}\times(\frac{1}{2}\sqrt{3})+10\times\frac{1}{3}\pi\right)$	M1	For use of limits
		$=\frac{1}{32}\left(3\sqrt{3}+\frac{10}{3}\pi\right)$	A1 **8**	For correct answer
			10	

[Turn over

8 **(i)**	$y = kx^2\,e^{-2x} \Rightarrow y' = 2kx\,e^{-2x} - 2kx^2\,e^{-2x}$ and	M1	For differentiation at least once
	$y'' = 2k\,e^{-2x} - 8kx\,e^{-2x} + 4kx^2\,e^{-2x}$	A1	For both y' and y'' correct
	$(2k - 8kx + 4kx^2 + 8kx - 8kx^2 + 4kx^2)e^{-2x} \equiv 2\,e^{-2x}$	M1	For substituting completely in D.E.
	Hence $k = 1$	A1 **4**	For correct value of k
(ii)	Auxiliary equation is $m^2 + 4m + 4 = 0 \Rightarrow m = -2$	B1	For correct repeated root
	Hence C.F. is $(A + Bx)e^{-2x}$	B1	For correct form of C.F.
	G.S. is $y = (A + Bx)e^{-2x} + x^2\,e^{-2x}$	B1√	For sum of C.F. and P.I.
	$x = 0,\ y = 1 \Rightarrow 1 = A$	M1	For using given values of x and y in G.S.
	$y' = B\,e^{-2x} - 2(A + Bx)e^{-2x} + 2x\,e^{-2x} - 2x^2\,e^{-2x}$	M1	For differentiating the G.S.
	$x = 0,\ y' = 0 \Rightarrow 0 = B - 2A \Rightarrow B = 2$	M1	For using given values of x and y' in G.S.
	Hence solution is $y = (1 + x)^2\,e^{-2x}$	A1 **7**	For correct answer
(iii)	$\dfrac{d^2y}{dx^2} = 2 - 4 = -2$ when $x = 0$	B1	For correct value -2
	Hence $(0, 1)$ is a maximum point	B1	For statement of maximum at $x = 0$
	$\dfrac{dy}{dx} = 2(1 + x)e^{-2x} - 2(1 + x)^2\,e^{-2x} = -2x(1 + x)e^{-2x}$,		
	so there are no turning points for $x > 0$	M1	For investigation of turning points, or equiv
	Hence $0 < y \leqslant 1$, since $y \to 0$ as $x \to \infty$	A1 **4**	For complete proof of given result
		15	